Al-Azhar University Civil Engineering Research Magazine (CERM)
Vol. (39) No. (4) October, 2017

ELASTO-PLASTIC ANALYSIS OF MINDLIN PLATE WITH LAYERING APPROACH

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الملخص:

التحليل الأنشائى المرن – اللدن للمنشآت اللوحيه يعتبر مهم فى العديد من التطبيقات الأنشائيه. تحليل الأنهيار اللدن للمنشآت يمكن أن يتم باستخدام نظريه الحدود المعروفه والتى توضح أن الحلول الدقيقه للصفائح غير ممكنه. عامه حمل الأتهيار المحسوب للألواح يقع بين حدين معتمدا على الطريقه المستخدمه. الحل بأستخدام الحد الأعلى لحمل الأنهيار يمكن الحصول عليه بالتحليل بأستخدام طريقه خطوط الأنكسار. والحلول المرنه للألواح يمكن أعتبارها كحد أدنى لقيمه حمل الأنهيار المرن. طريقه العناصر المحدده غير الخطيه تعطى للمهندس الأنشائى القدره على حل المشاكل المعقده بطريقه واقعيه قريبه للحل الحقيقى ومشتمله على الشكل الهندسى وحاله الركائز للمنشأ. تم عمل برنامج كمبيوتر بواسطه الباحث مستخدما لغه الفورتران. برنامج الكمبيوتر مبنى على التحليل اللاخطى فى المواد والشكل الهندسى ومستخدم عنصر يصلح للألواح النحيفه والسميكه وبأستخدام عنصر أنحناء Heterosis ونظريه Reissner-Mindlin للصفائح. وكذلك تطبيق صيغه الحل بأستخدام طريقه الطبقات وعمليه التكرار فى حل المعادلات اللاخطيه. يتم مقارنه القيم العدديه من برنامج الكمبيوتر بالحد الأدنى و الأعلى بأستخدام التحليل باستخدام المعادلات التفاضليه. القيم العدديه لبعض التطبيقات الشائعه توضح أن النتائج دقيقه وكذلك كفاءه النموذج الرياضى المستخدم فى برنامج الكمبيوتر المعد.

ABSTRACT

The elasto-plastic analysis of plate structures is of considerable interest in various areas of structural applications. Plastic collapse analysis of a structure can be carried out by the well known bound theories which indicate that exact solutions for plates are not always possible. In general, the calculated collapse load lies between the two limits depending on the approach adopted. Upper bound solution for collapse load can be obtained from yield line analysis of plates. Any elastic solution of the plates can be considered as a lower bound solution for the plastic collapse load. Nonlinear finite element method of analysis provides the structural engineer with capabilities to analyze very complex problems of engineering in much more realistic manner including geometry and support conditions that come closer to the reality. A software program called EPALPE (Elasto-Plastic Analysis for Layered Plate Element) is prepared by the author using Fortran 77. The software is used for elasto-plastic and geometrically nonlinear analysis of thin and thick plate structures using the Heterosis plate bending element based on Reissner-Mindlin plate theory. The formulation utilizes discrete layered approach and incremental-iterative algorithm to solve the system of nonlinear equations. The numerical results from the program EPALPE have been compared with the lower and upper bound solutions from analytical solutions. The numerical results for some bench mark applications demonstrate the acceptable accuracy and efficiency for the computational model which is implemented in the prepared computer program.

Keywords: Elasto-Plastic Analysis, NonLinear Finite Element Method, Reissner-Mindlin Plate Theory, Layered Approach, Lower and Upper Bound, Elasticity Theory, Yield Line Theory.

INTRODUCTION

The analysis of elasto-plastic bending problems of the plates has been performed by many researchers [1], [2], [3], [4]. The upper and lower bounds ultimate capacities of the plate can be carried out by the theorems of limit analysis. The upper bound theorem (unsafe theorem) states that if a load is found which corresponds to any assumed collapse mechanism, then the load must be equal to or greater than the true collapse load. Yield line analysis of plates is simply application of the upper bound theorem to a plate collapse mechanism [5]. The lower bound theorem (safe theorem) states that if for any load a stress distribution can be found which both satisfies all equilibrium conditions and nowhere violates yield conditions, then the load cannot cause collapse . Any elastic solution is a lower bound for the plastic collapse load [6]. Once an upper bound for collapse is obtained through the yield line theory, the exact collapse load can be bracketed if a lower bound solution is known. The non linear finite element method is now firmly accepted as a most powerful general technique for the numerical solution of a variety of problems encountered in engineering [3], [4], [7], [8], [9], [10]. In this work, the computational model involves many considerations including providing a layered plate bending element based on Reissner-Mindlin plate theory [11]. The 9 node Heterosis element which was developed by Hughes et al. [3], [9] is introduced, elasto-plastic material responseand geometrically nonlinear is considered in this model. In the layered model, a plate is divided into layers of different thickness where stresses are calculated and the yield condition is checked for each layer separately. The forces and moments are then calculated by integration through the thickness. To implement the elasto-plastic material behavior three requirements have to be met a) The elastic constitutive relation must be formulated to describe material behavior under elastic condition b) A yield criterion indicating the stress level at which plastic flow commences must be postulated c) A relationship between stress and strain must be developed for post yield behavior i.e. when the deformation is made up of both elastic and plastic components. There are different numerical procedures that can be incorporated in the solution of nonlinear problems. A successful procedure must include incremental / iterative method to solve the governing equation and convergence criteria or termination schemes to end the solution process with the acceptable accuracy [8], [9].

OBJECTIVES OF PRESENT STUDY

The objectives of this paper can be summarized as follows:

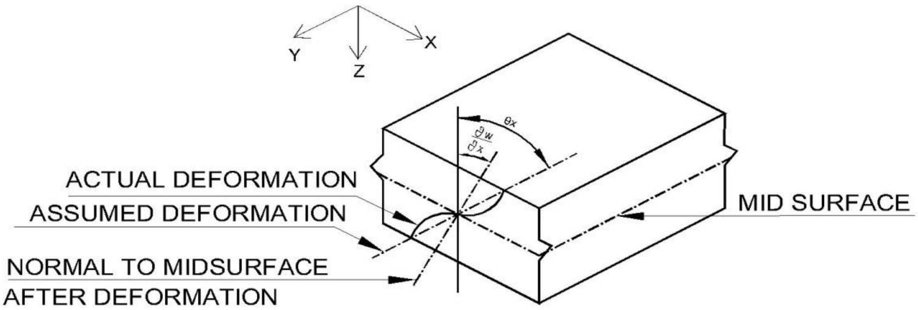
1- To prepare and develop a computer program EPALPE for the elasto-plastic and geometrically nonlinear analysis of plate in bending .
2- To review and examine the computational model for elasto-plastic analysis of Heterosis layered plate bending element based on Reissner-Mindlin plate theory.
3- To compare the numerical results from EPALPE for some bench mark problems with the analytical results from yield line and elasticity theories of plates which are considered as upper and lower bounds solution.

THE COMPUTATIONAL MODEL

The computational model includes the following:-

1) The plate bending element based on Reissner-Mindlin plate theory as per the following assumptions :-
    a) Displacements are small compared with the plate thickness.
    b) The stress normal to the mid-surface of the plate is negligible.
    c) The normal to the mid-surface before deformation remains straight but not necessarily normal to the mid-surface after deformation as shown in Fig.1.

Fig.1 Mindlin Plate Theory Assumptions



2) The quadratic 9 node Heterosis element used in this work

The 9 node Heterosis quadrilateral element exhibits improved characteristics in comparison with 8 node Serendipity and 9 node Lagrangian [8], [9] to analyze thick or thin plates . The Heterosis element is formulated using 9 node Lagrangian shape functions for rotations (θ_x , θ_y) and 8 node Serendipity shape functions for lateral displacement (W) as shown in Fig. 2. It was assumed that the geometry and displacement vary quadratically over the element by the following shape function.

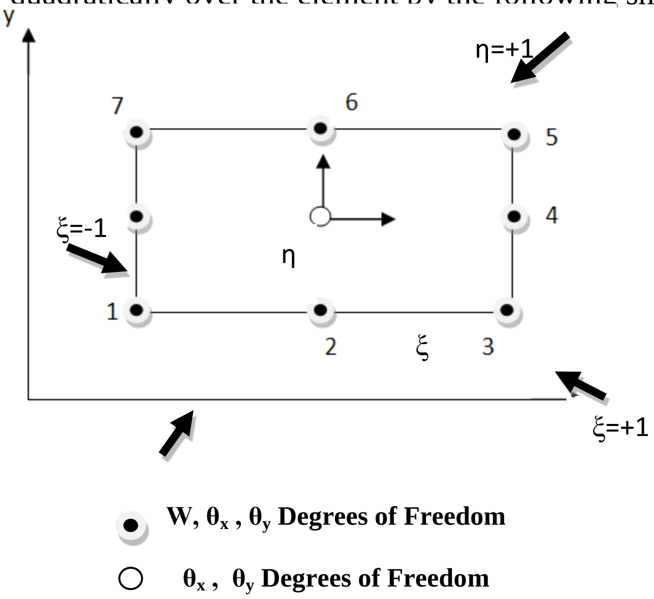


Fig.2 Natural Coordinate Nine Node Heterosis Element

For corner nodes :

$N_i(\xi , \eta) = \frac{1}{4} (1+ \xi \xi_i) (1+ \eta \eta_i) (\xi \xi_i + \eta \eta_i -1)$ i=1,3,5,7

(1-a)

For mid-side points :

$N_i(\xi , \eta) = \frac{\xi_i^2}{2} (1+ \xi \xi_i) (1 - \eta^2) + \frac{\eta_i^2}{2} (1+ \eta \eta_i) (1 - \xi^2)$ i=2,4,6,8

(1-b)

For central node :

$N_i(\xi, \eta) = (1 - \xi^2)\ (1 - \eta^2)$ $\qquad$ i = 9
(1-c)

3) The layered approach

In the discrete layered approach the plate is divided into a series of layers of different thickness and material. Each layer contains stress points on its mid-surface. The stress components of the layer computed at these stress points are assumed to be constant over the thickness of each layer, so that the actual stress distribution over the plate thickness is modeled by a piecewise constant approximation as seen in Fig.(3).

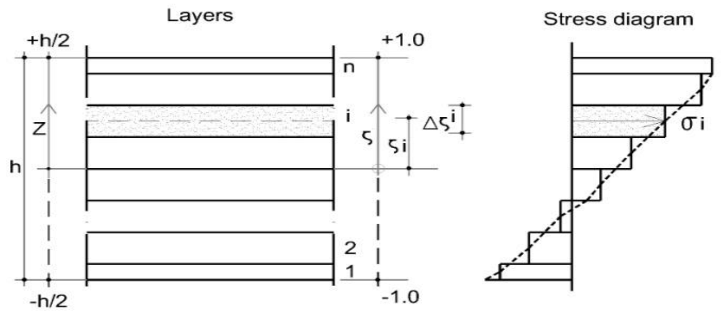


Fig.(3) Layered Subdivision of Plate and the Corresponding Stress

In the present work the strain matrix $|B|$ is calculated at mid-surface of each layer, the element stiffness matrix $|K|$ and the internal force vector { P }are thus defined as follows:

K = $\int_v B^T D\, B\, dV$

P = $\int_v B^T \sigma\, dV$

4) The constitutive relation for elasto-plastic plates has three steps

a) The elastic constitutive relation based on the generalized Hooke's law.
b) The yield criterion indicating the stress level at which plastic flow commences based on the generalized Huber-Mises law.
c) The relation between the incremental stress and incremental strain after the onset of initial yielding.

The incremental stress – strain relationships for a plate layer may be expressed by

$\{\Delta\sigma\}_{5x1} = [D]_{5x5}\{\Delta\varepsilon\}_{5x1}$ (2)

Taking into consideration the assumption of zero stress in the direction perpendicular to the plate mid surface $\sigma_3 = 0$

$[D]_{5x5}$ is the elasticity matrix which is calculated for the material axes

$$\begin{Bmatrix} \Delta\sigma_1 \\ \Delta\sigma_2 \\ \Delta\tau_{12} \\ \Delta\tau_{13} \\ \Delta\tau_{23} \end{Bmatrix} = \begin{bmatrix} \frac{E_1}{(1-\upsilon_{12}\upsilon_{21})} & \frac{E_2\upsilon_{12}}{(1-\upsilon_{12}\upsilon_{21})} & 0 & 0 & 0 \\ \frac{E_2\upsilon_{12}}{(1-\upsilon_{12}\upsilon_{21})} & \frac{E_2}{(1-\upsilon_{12}\upsilon_{21})} & 0 & 0 & 0 \\ 0 & 0 & G_{12} & 0 & 0 \\ 0 & 0 & 0 & k_1G_{13} & 0 \\ 0 & 0 & 0 & 0 & k_1G_{23} \end{bmatrix} \begin{Bmatrix} \Delta\epsilon_1 \\ \Delta\epsilon_2 \\ \Delta\gamma_{12} \\ \Delta\gamma_{13} \\ \Delta\gamma_{23} \end{Bmatrix} \qquad (3)$$

Where

The subscript 1 , 2 and 3 refer to the directions of the three principal axes of anisotropy

The σ,s and τ,s are stress components

The ϵ,s and γ,s are strain components
E is the Young,s modulus
ν is the Poisson,s ratio
k1 , k2 are shear correction factors in the $\overline{13}$ and $\overline{23}$ planes

The yield criterion can be expressed for an anisotropic material in a similar manner for isotropic material
$\overline{\sigma} = [\alpha_{12}(\sigma_1-\sigma_2)2 + \alpha_{23}(\sigma_2-\sigma_3)2 + \alpha_{13} (\sigma_3-\sigma_1)2 + 3\alpha_{44}\, \tau_{12} + 3\alpha_{55}\, \tau_{13} + 3\alpha_{66}\, \tau_{23}]^{0.5}$ (4)
Where $\bar{\sigma}$ is effective stress and α's are parameters of anisotropy
By using the assumption $\sigma_3 = 0$
$\overline{\sigma^2} = a_1\, \sigma_1^2 + 2a_{12}\, \sigma_1\, \sigma_2 + a_2\, \sigma_2^2 + a_3\, \tau_{12} + a_4\, \tau_{13} + a_5\, \tau_{23}$ (5)
Where a_1, a_{12}, a_2, a_3, a_4 and a_5 are anisotropic parameters which can be determined experimentally.

5) Geometry Nonlinear

Total lagrangian formulation is adopted in which large deflection and moderate rotations (in the sense of the Von Karman hypotheses) are accounted for
Strain-Displacement matrix B is separated into the usual part B and nonlinear contribution part BL. So that
$B = B + BL$ (6)
Consequently, geometric stiffness matrix $K\sigma$ may be defined as
$K\sigma = da \int_v dB^T\, \sigma\, dV$ (7)
Then, use of (2) and (12)
$K = K + K\sigma$ (8)
Where K is the total stiffness matrix

6) Incremental/iterative solution

In nonlinear analysis, the basic set of equilibrium equations to be solved at a certain load increment n is as following:
$\Psi_i^n = f_i^n - P_i^n = f^n - \int_v B^T\, \sigma_i^n dV \neq 0$ (9)
Where Ψ_i^n is the residual forces at iteration i, fn is the external applied loads and P_i^n is the internal equivalent forces at iteration i.
An iteration sequence must be performed for each load increment n in order to obtain a displacement field, a_i^n which provides a stress field σ_i^n in such that the residuals Ψ_i^n vanish. In particular, the displacements are updated at each iteration according to
$a_i^n = a_{i-1}^n + \Delta a_i^n$ (10)
Where Δa_i^n denotes the displacement change occurring during the iteration.
$\Delta a_i^n = [K_{i-1}^n]^{-1}\ \Psi_{i-1}^n$ (11)
In which
$[K_{i-1}^n]^{-1}$ is the tangential stiffness matrix of the structure evaluated at the beginning of the i^{th} iteration
In an incremental iterative solution strategy, the solution obtained at the end of each iteration is checked to see whether it has converged within preset tolerance or whether it is diverging.

The iteration displacements Δa_i at iteration i are used to monitor convergence of nonlinear solution. for convergence the norm of the incremental displacements is required to be less than a specified percentage of the norm of the total displacements (a_i).

NUMERICAL APPLICATIONS

In order to compare the numerical results obtained by the program EPALPE with analytical results from yield line and elasticity theories of plates to confirm the validity, convergency, accuracy and efficiency, numerical solutions for three specific problems with a variety of end condition (simply supported SSSS, clamped edge CCCC and two opposite edges clamped and the others simply supported SCSC) are presented. All three problems involve isotropic square plates subjected to lateral loads that are uniformly distributed throughout the plate. Square plate will be analyzed and by taking advantage of symmetry a 3x3 mesh of Heterosis elements is adopted in a symmetric quarter as shown in Figs.4, 5 and 6.

Physical and geometrical parameters for these plates are as following:-

Youngs' modulus (E) = 30000 N/mm^2

Shear modulus (G) = 11540 N/mm^2

Yield strength of plate material (σ_y) = 30 N/mm^2

Poissons' ratio (ν) = 0.3

Plate thickness (t) = 0.2 m

Square plate side length (L) = 6 m

The flexural rigidity of the plate (D) = $\frac{E\,t^3}{12(1-\nu^2)}$

The fully plastic moment (M_p) = $\sigma_y \frac{t^2}{4}$

Table 1 shows the numerical factor $\frac{q\,L^2}{M_P}$ from analytical analysis using elasticity and yield line theories as given in references [3], [4], [7] for square plates subjected to uniform load (q) and of different boundary conditions.

Table 1: Numerical Factor $\frac{q\,L^2}{M_P}$ From Analytical Analysis of Square Plates Subjected to Uniform Load Using Elasticity and Yield Line Theories

Boundary condition	CCCC	SSSS	SCSC
Yield line theory	48.00	24.00	36.00
Elasticity Theory	43.30	20.88	30.12

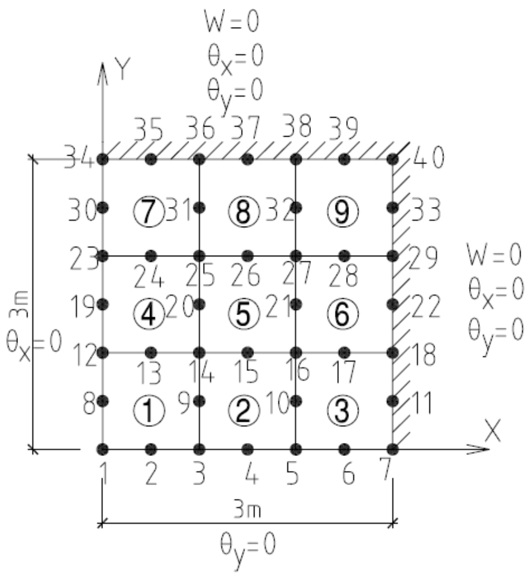


Fig. (4) A Quarter of Clamped Square Plate Discretized into mesh 3x3 using Heterosis element

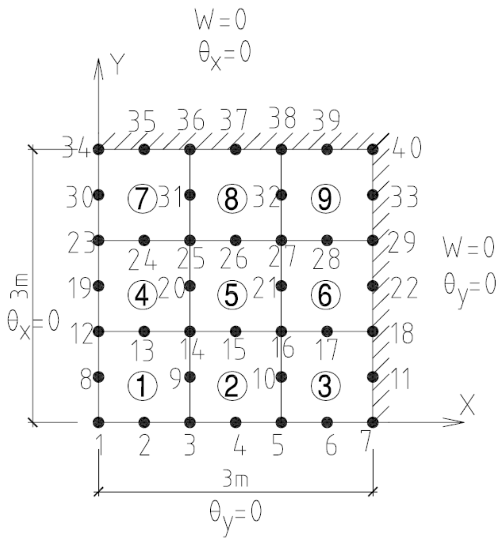


Fig. (5) A Quarter of Simply Supported Square Plate Discretized into mesh 3x3 using Heterosis element

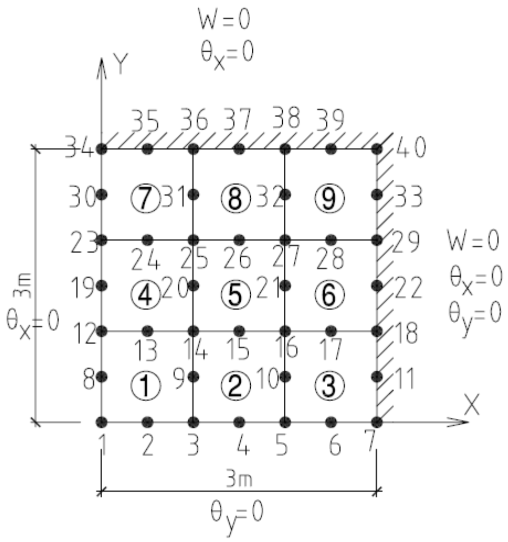


Fig. (6) A Quarter of Two Opposite Edges Clamped and Other One Simply Supported Square Plate Discretized into mesh 3x3 using Heterosis element

RESULTS AND DISCUSSION

Figs. 7, 8 and 9 show the load-deflection curve for a square plate subjected to uniform load with different types of end conditions. Where w is the central plate deflection.

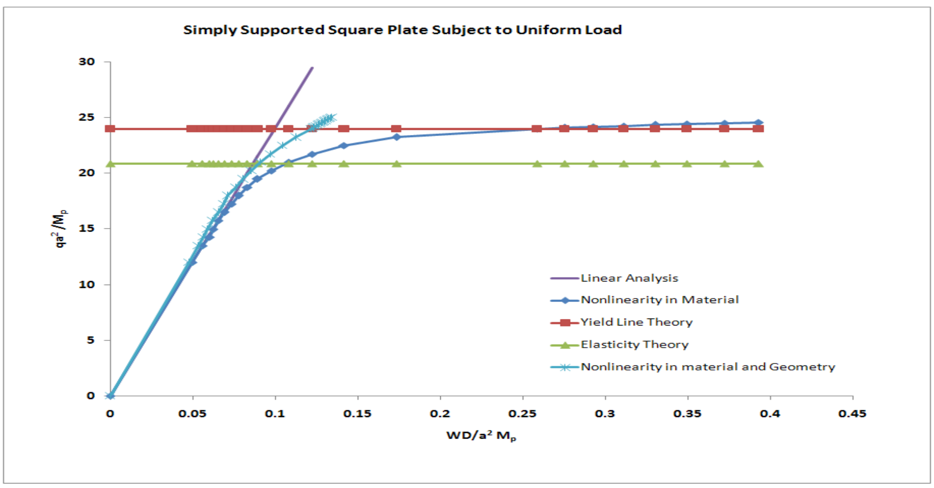


Fig. (7) Load-Deflection Curve for Simply Supported Square Plate Subjected to Uniform Load

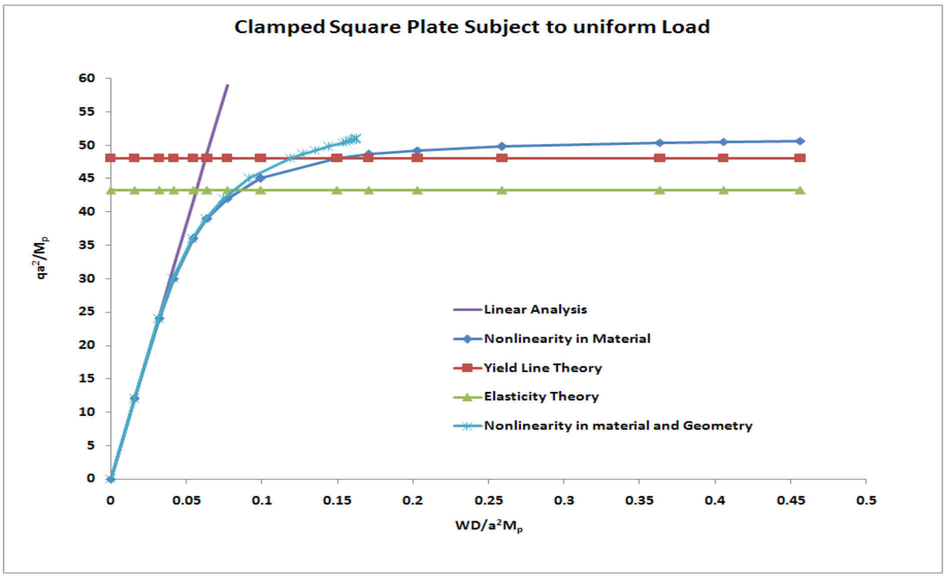


Fig. (8) Load-Deflection Curve for Clamped Square Plate Subjected to Uniform Load

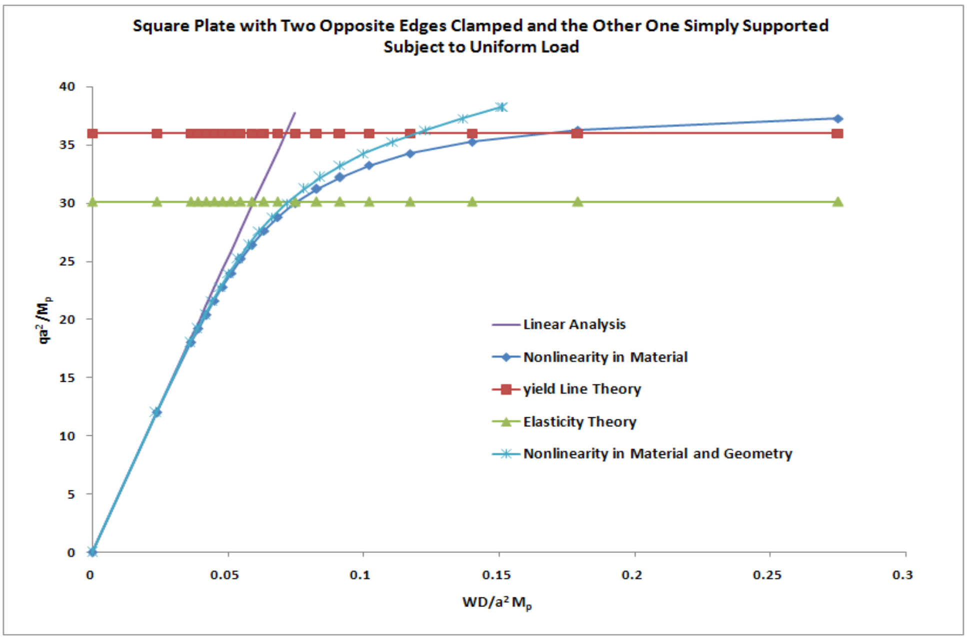


Fig.(9) Load-Deflection Curve for Square Plate with Two Opposite Edges Clamped and Other Ones Simply Supported Subjected to Uniform Load

CONCLUSIONS

The main conclusions of the work described in this paper are summarized as follows:

1- By utilizing the present computational model which is implemented in the computer program EPALPE the elasto-plastic problems for the plates can be treated with reasonable accuracy.

2- The applications were solved for isotropic plate as special case from the anisotropic plate gave good accuracy.

RECOMMENDATIONS AND FUTURE WORK

Listed below are the recommendations for future work based on the work done in this paper:

1- The composite plates can be analyzed by utilizing the elasto-plastic analysis for anisotropic plates with layered approach.
2- The computational model can be developed to analyze the reinforced concrete slabs taking into consideration the real material behavior and the nonlinearities in material and geometry.
3- The program EPALPE can be developed to analyze reinforced concrete slab including the real material behavior of concrete and steel reinforcement.

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